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Strength of Cylinders

Designing pressure containers for chemical reactions is complicated by such factors as corrosion, high temperatures, and fluctuating loads. Many problems can be approached with conventional methods, but fatigue and creep require special treatment.

Design for cylinders is discussed generally, but it is believed that these principles can be applied to other shapes and conditions. A rapid method of assessing strain and stresses in autofrettage cylinders is given along with a means of adapting this to the creep problem. Results of fatigue tests with repeated pressure applications are also reviewed.

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HIGH PRESSURE

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Strength of Cylinders

New design methods are described here for meeting problems of fatigue and creep imposed by higher temperatures and fluctuating loads

THE CHEMICAL INDUSTRY has greatly increased the importance of correct design for pressure containers. The combined effects of such factors as corrosion, high temperatures, and fluctuating loads have raised the demands beyond those which can be met by straightforward interpretation of ordinary theoretical methods. Unfortunately, with problems of this kind, designers have too often resorted to empirical formulas of doubtful validity, sometimes with disastrous consequences. Nevertheless, many of these difficulties can be resolved by slight extensions and modifications of conventional methods. Although only cylinders are considered here, it is believed that these ideas can be applied to other shapes and conditions.

General Considerations

Figure 1A shows a section through half of a cylinder subjected to an internal pressure of P_i and an external pressure of P_o. The suffix $_o$ represents conditions at the external surface and $_i$ those at the internal, and for stresses and strains, suffixes $_r$, $_t$, and $_a$ represent those in the radial, tangential or hoop, and axial directions, respectively. Figure 1A shows that the total force tending to push a unit axial length of the cylinder up is $2r_iP_i$. A similar downward force of $2r_oP_o$ arises from the external pressure. The difference between these must be balanced by the tangential stress in the cylinder wall, and because this is not normally constant across the section, the general condition of stability must be

$$2\int_{r_i}^{r_o} f_t\,dr = P_iD_i - P_oD_o \qquad (1)$$

where D_o and D_i are external and internal diameters, respectively.

When wall thickness (t) is small compared with diameter, the tangential stress across it is virtually constant, and neglecting the external pressure, Equation 1 reduces to

$$f_t = \frac{P_iD_i}{2t} \qquad (2)$$

which is the commonly used "thin tube formula."

The section in Figure 1A can also be regarded as a thin elemental ring of a larger, thicker cylinder by writing r and $r + \delta r$ for r_i and r_o, and f_r and $f_r + \delta f_r$ for P_i and P_o. The tangential stress can then certainly be considered constant across the thin radial extent of δr. It is convenient in subsequent analysis to consider all stresses as tensile, whereupon those that are compressive will appear with a negative sign. Thus, Figure 1A can be transposed as shown in Figure 1B, and the equation of static equilibrium simplifies in the limiting condition to

$$f_t - f_r = r\frac{df_r}{dr} \qquad (3)$$

Since f_t and f_r are principal stresses, their difference is twice the maximum shear stress in that plane. Normally f_t is tensile and f_r compressive with the axial stress usually much smaller numerically than either; therefore, the two former lie in the plane of greatest shear stress. If S is this shear at radius r, then

$$f_t - f_r = 2S$$

and

$$2S = r\frac{df_r}{dr} \qquad (4)$$

which can be integrated to give

$$P_i - P_o = 2\int_{r_i}^{r_o} (S/r).dr \qquad (5)$$

These considerations have assumed only geometrical symmetry; therefore, the equations derived are true for any material, elastic, or plastic or of any intermediate condition. This becomes important because any cylinder problem can be solved with Equation 4 if distribution of shear stress across its section is known.

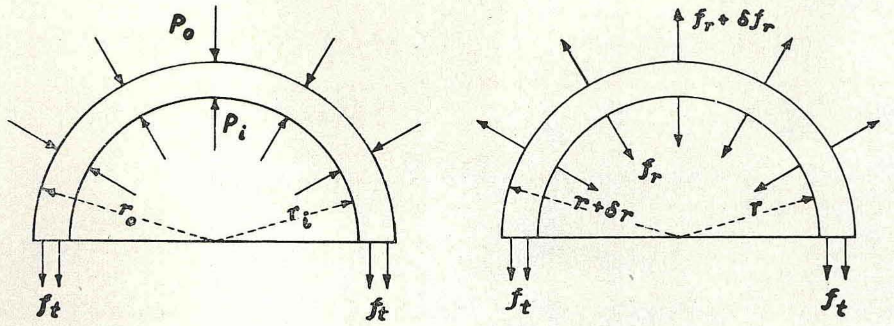


Figure 1. Section of half a cylinder

A. Subjected to internal and external pressures P_i and P_o, respectively

B. When considered as a thin elemental ring of a larger, thicker cylinder

Elastic Conditions

This is the one case that can be completely solved analytically, and the resulting solution is of direct utility. It is generally called Lamé who was probably the first to evolve it. The ensuing analysis is dealt with in most textbooks (*16*, *26*) and leads to the following general results for the stresses

$$f_t = \frac{-P_o r_o^2 + P_i r_i^2 - (P_o - P_i) r_o^2 r_i^2 / r^2}{r_o^2 - r_i^2} \qquad (6)$$

$$f_r = \frac{-P_o r_o^2 + P_i r_i^2 + (P_o - P_i) r_o^2 r_i^2 / r^2}{r_o^2 - r_i^2} \qquad (7)$$

The value of f_a is indeterminate without further knowledge of the end conditions, but it must be constant across the section. For a cylinder with end covers attached to the walls so that the end load has to be carried by the axial stress in the cylindrical portion, it is given by

$$f_a = \frac{-P_o r_o^2 + P_i r_i^2}{r_o^2 - r_i^2} \qquad (8)$$

On the other hand, in the cylinder of a simple hydraulic press, end loads are taken by the frame or columns and the axial stress is zero except for small forces arising from friction of the plunger.

A further important relation is shear stress in the cross-sectional plane, which is half the algebraic difference between f_t and f_r—i.e.,

$$S = \frac{(P_o - P_i) r_o^2 r_i^2 / r^2}{r_o^2 - r_i^2} \qquad (9)$$

f_t, f_r, and S have their greatest values at the inside surface. For most practical cases, external pressure can be neglected, and the formulas for maximum stresses set up by an internal pressure, P, are greatly simplified. Thus

$$f_t(\text{max.}) = P\frac{K^2 + 1}{K^2 - 1} \qquad (10)$$

$$f_r(\text{max.}) = -P \qquad (11)$$

$$S(\text{max.}) = P\frac{K^2}{K^2 - 1} \qquad (12)$$

Here K is the ratio, r_o/r_i, usually called the diameter ratio.

This last simplification shows incidentally that these formulas involve only the ratio of the diameters or radii and not their absolute size. Thus, the principle of similarity applies and the theory gives no hint of scale factor.

This analysis can be checked with reasonable accuracy because it permits the calculation of the external strains caused by a given pressure inside a cylinder. Thus, the increase in external diameter, x, is

$$x = \frac{P}{E} \cdot \frac{D_o(2 - c)}{K^2 - 1} \qquad (13)$$

where E and c are Young's modulus and Poisson's ratio, respectively, and the over-all axial strain, y, for a vessel of L length is

$$y = \frac{P}{E} \cdot \frac{L(1 - 2c)}{K^2 - 1} \qquad (14)$$

In each case these refer to vessels which support their own end loads.

In an actual test on a reaction vessel done in 1939, the dimensions were 80 inches in over-all length and $7^1/_2$ and $11^3/_4$ inches for internal and external diameters, respectively. The material was a heat-treated nickel-chrome-molybdenum steel having, in pounds per sq. inch, an ultimate strength of approximately 125,000, a yield of 103,000, and Young's modulus and Poisson's ratio of approximately 28.0×10^6 and 0.3, respectively (Table I). Closer agreement cannot be expected, especially because the formulas involve Poisson's ratio, which is not usually known with great accuracy.

Application of the theory to design

Table I. Strains in Actual Pressure Vessel

(At 34,000 lb./sq. in.)

Dimensional Change	Inches Calculated	Inches Observed
Expansion of external diameter	0.0165	0.0170
Longitudinal extension	0.0265	0.0275

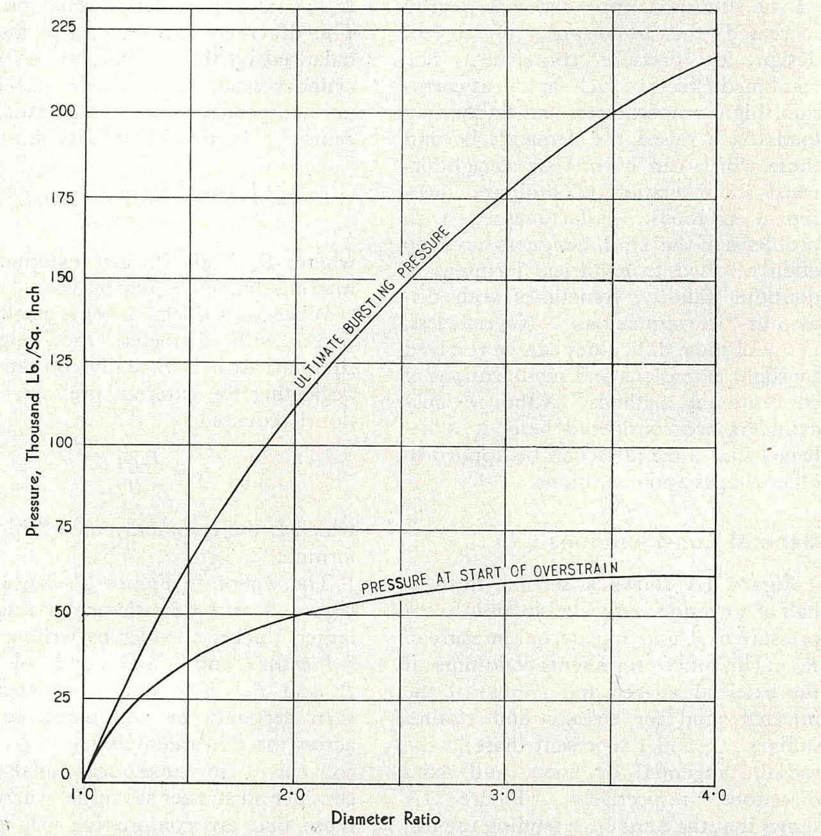


Figure 2. Point where overstrain begins, compared with bursting pressure for a cylinder of alloy steel having a tensile strength of 137,000 pounds per square inch

problems needs further consideration, but even if some serious limitations are revealed, there is little doubt that the method can answer all questions arising from the intended use of cylinders for static pressures at temperatures below creep conditions.

Stress Distribution in Elastic Cylinders

Figure 3A shows the variation of radial, tangential, and shear stresses across the section of a cylinder whose diameter ratio is 4.5. Intensity of each rises abruptly to its peak value at the inner surface. Equations 6 and 7 also show that both steepness of the peak and difference between the highest and lowest values decrease rapidly as the wall gets thinner; consequently, for small values of K, the thin tube formula of Equation 2 can be safely used.

Several other simplified formulas are also used from time to time, such as the "outside diameter formula" and the "mean diameter formula" which, as their names imply, consider pressures acting over a width equal in the first case to the outside diameter and in the second to the mean diameter. The corresponding relations are, respectively,

$$f_t = \frac{PD_o}{2t} \tag{15}$$

and

$$f_t = \frac{P(D_o + D_i)}{4t} \tag{16}$$

mean diam formulas

Table II below shows the ratio of f_t to pressure for each case. Thus, the thin tube formula can be used safely up to a diameter ratio of 1 to 1.1—i.e., when the wall thickness is 5% of the bore diameter, but above this, the error exceeds 5% and gets rapidly worse as the wall is thickened. Moreover, the error is on the side of danger.

The outside diameter formula departs from the correct values initially at about the same rate, but it has the merit of erring on the side of safety. The mean diameter formula, however, is considerably better—less than 4% off for a thickness equal to 25% of the bore ($K = 1.5$). But other considerations are needed in that range, and none of the simplified theories should be used for cylinders exceeding a diameter ratio of 1 to 1.1.

It is important that shear stress in a thick cylinder (Figure 3A) rises almost to the same value as the maximum tensile stress. The reason for this, of course, is that the other principal stress in that plane is compressive. By comparison with a tensile test—most design work is still related to tensile tests—the ratio of shear to direct stress is much higher; in the tensile test it is only 50%. This implies that shear is probably a better basis for cylinder design.

Table II. Comparison of Various Formulas

Diameter Ratio	Ratio Thickness to Bore, %	f_t/P by Thin tube formula	Outside diameter formula	Mean diameter formula	Exact formula[a]
1.01	0.5	100.0	101.0	100.5	100.5
1.05	2.5	20.0	21.0	20.5	20.51
1.10	5.0	10.0	11.0	10.5	10.52
1.5	25.0	2.0	3.0	2.5	2.60
2.0	50	1.0	2.0	1.5	1.667
3.0	100	0.5	1.5	1.0	1.250
5.0	200	0.25	1.25	0.75	1.083

[a] f_t (max.)/P.

note

Application to Design

This problem is best illustrated by reference to one of the numerous papers describing careful tests of thick cylinders up to the points where they overstrain. For instance, Cook (*4*) tested a series of cylinders of a steel containing 0.21% carbon and detected onset of overstrain by sensitive mirror extensometers. Tensile tests (mean of three tests in each case with a maximum deviation of about 1% from the mean) showed the material to have upper and lower yield stresses of 52,300 and 38,300 pounds per sq. inch, respectively. Actual plots of diametral expansion against internal pressure as given by Cook show fairly sharp deviations from linearity which allow critical pressures to be estimated with an accuracy of at least 2%.

Column 4 of Table III shows that maximum tensile stress cannot be the criterion of overstrain in these cylinders. The maximum shear stresses, however, are more consistent and range from 28,200 to 30,060 with a mean value of 29,300—i.e., a variation of 2.6% above and 3.7% below the mean.

The maximum shear stress in the tensile specimens at their upper yield points was, however, 26,150 pounds per sq. inch; thus, direct comparison of the maximum shear stress in the two types of loading shows a deviation of more than 10%. However, theoretical reasoning as well as an increasing volume of experimental evidence supports the shear-strain-energy hypothesis. This is usually associated with the names of von Mises, Hencky, or Huber, but Sopwith and Morrison (*24*) have recently shown that it was first propounded by Clerk Maxwell in 1856. According to this, overstrain of a cylindrical wall occurs when maximum shear stress reaches a value of $1/\sqrt{3}$ times the tensile upper yield stress; thus, yield in these cylinders would be expected when shear stress reached 30,200 pounds per sq. inch. The correlation is still not perfect, but the difference from the mean is now less than $2^1/_2$%. Because this is within the range of scatter for the cylinder results themselves, it is the best of the criteria so far put forward.

Therefore, the most satisfactory basis of design is arranging for the maximum shear stress of the cylinder, as given by Equation 12, to equal (when raised if required by an appropriate safety factor) the tensile yield stress divided by $\sqrt{3}$. Thus, if the safety factor is λ and the tensile yield stress (or selected proof stress) is f_y, the required diameter ratio is

$$K = \sqrt{\frac{f_y}{f_y - \lambda P\sqrt{3}}} \tag{17}$$

For example, if a pressure of 1000 atm. (14,700 pounds per sq. inch) has to be contained with the material of Cook's cylinders, allowing a safety factor of 1.5, what will the necessary diameter ratio be? Here $f_y = 52{,}300$; $P = 14{,}700$; $\lambda = 1.5$. Thus, $K = 1.93$. Consequently, if a 6-inch bore were required, the outer diameter would have to be just over $11^1/_2$ inches. The pressure could then rise 50% above its designed working value before the vessel became overstrained.

Table III. Results of Pressure-Dilatation Experiments

Dia. Ratio	Int. Dia., In.	At Onset of Yield, Lb./Sq. In. Pressure	Max. tang. stress	Max. shear stress
1.168	0.750	7,560	49,280	28,670
1.167	0.750	7,820	51,070	29,610
1.50	0.750	15,680	40,760	28,220
2.0	0.375	21,950	36,470	29,280
3.0	0.259	26,620	33,350	29,970
4.0	0.250	28,220	32,000	30,060

Limitations of Simple Elastic Cylinders

Having established that the most probable criterion of elastic breakdown in a cylinder is the shear-strain-energy hypothesis, it is desirable to take stock of the position as revealed by Equation 12. If no safety factor is introduced, even when the cylinder is infinitely thick, there is a limiting condition where shear stress becomes numerically equal to pressure. In the example considered here, the absolute limit of pressure that could be contained without overstrain by a simple cylinder of this material is less than 30,200 pounds per sq. inch.

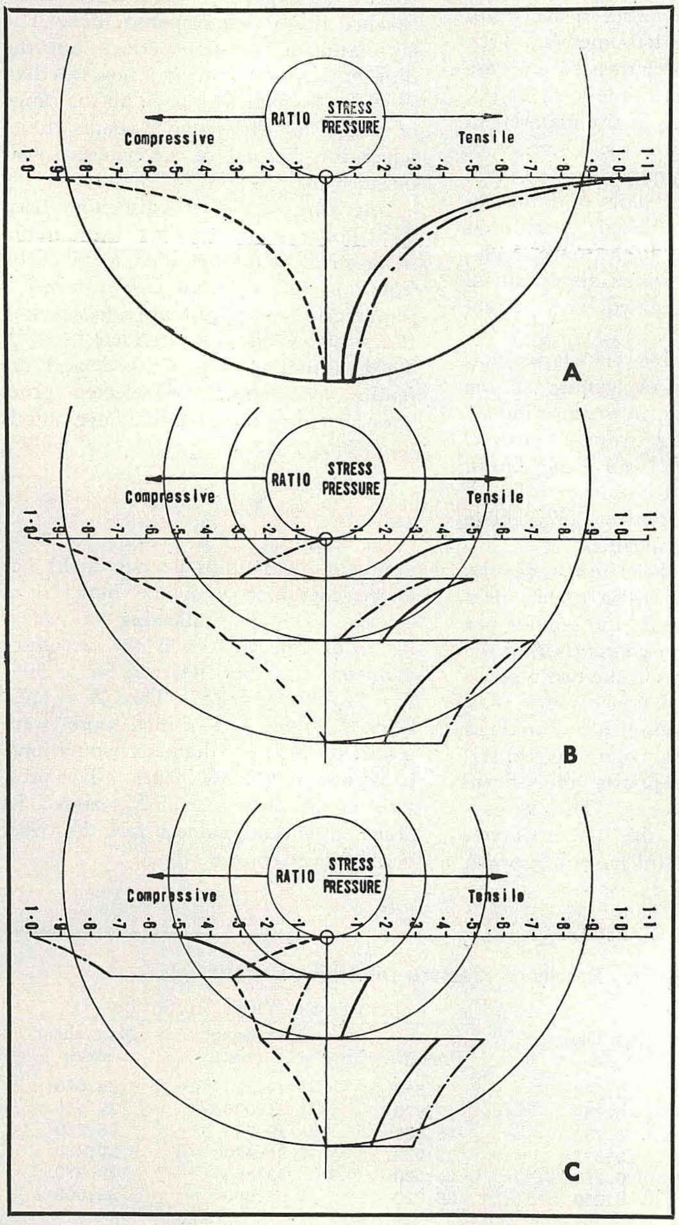
This is a serious limitation but fortunately there are several ways to overcome these restrictions. The key is provided by Figure 3A which shows that only a small proportion of the cylinder wall is highly stressed. Clearly, if a way could be found of making the outer layers take a fairer share of the load, the pressure contained could be greatly increased. When a cylinder of ductile material is loaded to the point where overstrain begins, it may require more than twice that pressure to burst it if the cylinder is thick (Figure 2); this again suggests ways of overcoming this limitation.

Reinforcement of Elastic Cylinders

Increasing the load taken by the outer layers can be done in a number of ways. All involve the same basic principle—namely, reinforcing a core tube either by shrinking other tubes over it, or by winding around it layers of continuous wire or strip. In this way, a residual tensile stress is developed in the outer layers and a corresponding compression in the core. And, when the pressure load is applied, compression in the core cancels out some resulting tensile stressing at the expense of increased tension in the lightly stressed outer layers.

As previously stated, these problems ought to be considered from the standpoint of shear stress. Thus, the core tube would be sheared in one direction of rotation by the reinforcing layers, and in the reverse direction by the pressure load. Shear in the layers, on the other hand, is in the same direction from both causes, and the effect is therefore additive in the latter and partially counteracting in the former.

Reinforcement by shrinking is susceptible to accurate analytical treatment, and the resulting equations indicate ultimate limits of all reinforcing techniques. Here, the analysis is fortunately simplified if the problem is considered from the standpoint of shear, because, as is seen in Equation 9, shear stress, unlike the radial and tangential stresses, is a function of the difference of pressures applied at the surfaces and is independent of their absolute values. Maximum shear stress in any cylinder from Equation 9 simplifies to



◀ **Figure 3. Stresses for cylinders having a diameter ratio of 4.5 to 1**

A. Simple cylinder with internal pressure acting
B. Ideal triplex cylinder with internal pressure acting
C. Ideal triplex cylinder showing residual stresses
Stresses: shear ——— hoop — — — — — radial - - - - -

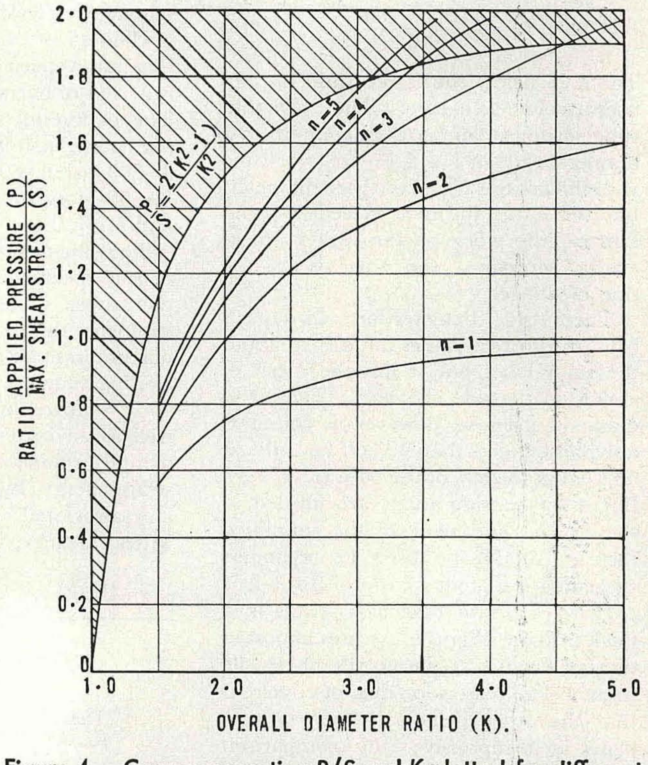


Figure 4. Curves connecting P/S and K plotted for different values of n with the curve of $P/S = \frac{2(K^2 - 1)}{K^2}$ superimposed

At any point in the shaded area, residual stress exceeds maximum working stress

$$S = \frac{(P_o - P_i)k^2}{k^2 - 1} \tag{18}$$

If now there are n components of which the first has a diameter ratio of k_1, the second k_2, and so on, a set of equations can be written for the shear stress in each, remembering that the contact pressure acting on the outside of one component is also that acting on the inside of the next. The solution of most problems of this kind then becomes a matter of suitably manipulating these equations.

The ideal case is where the maximum shear stress in each component has the same value because it will then be a minimum for the particular pressure, number of components, and over-all diameter ratio. It has been shown by this analysis (*13*) that contact pressures between successive components should decrease in arithmetic progression, and that contact radii should increase in geometric progression. If these conditions are substituted, a general equation is obtained in which S is given by

$$\frac{P}{S} = \frac{n(K^{2/n} - 1)}{K^{2/n}} \tag{19}$$

where K is the over-all diameter ratio of the composite cylinder. Figure 3B shows the stress distribution of a triplex cylinder of the same diameter ratio as the simple cylinder shown in Figure 3A; and Figure 3C shows residual stresses in the same cylinder when the internal pressure is removed.

The difference in dimensions needed to produce the required residual stresses on assembly can also be calculated. In fact the required shrinkage is given by the simple relation,

$$\frac{v}{r} = \frac{2P}{nE} \tag{20}$$

for each contact surface. That is, the same interference per unit of diameter is required for each component, although in practice, when there are more than two, the assembly of extra components over those already mated will require higher temperature differences because of strains in the latter.

Limits of Compound Construction

Equation 19 has the virtue of showing at a glance what might be achieved under extreme conditions—e.g., when the number of components and the over-all diameter ratio is large. Economic considerations are always likely to limit the number of components in ordinary shrink construction to 3 or 4, but other methods of reinforcement such as strip or wire winding and multi-layer construction can be regarded as essentially similar in principle and, therefore, capable of investigation by the same formula.

Equation 19 states that maximum shear stress in each layer is the same. Moreover, the thinner the layer, the less the difference between maximum and minimum shear stress; in other words it will be more even across the wall. Consequently, as the number of layers becomes great, the condition is approached where shear stress is constant throughout. It is then easy to integrate Equation 5 and show that

$$P = 2\,S \log_e K \tag{21}$$

This is in fact the limiting form of Equation 19 as n approaches infinity.

Equation 21 shows that theoretically an infinite pressure can be contained, if there is both infinite thickness and an infinite number of components. Also, the shrinkage required is then zero (Equation 20). This implies that the larger the number of components, the smaller the required shrinkage to reach the ideal condition of stress distribution. Moreover, the further condition of geometrical similarity of the components becomes less important as the number of layers becomes large; in fact it has been shown (*13*) that Equation 19 can be used to predict quite closely the elastic limit of cylinders constructed by the A. O. Smith Corp.'s multi-layer process. Vessels of this kind were tested to destruction under carefully controlled conditions by Jasper and Scudder (*8*) and their results conform well with this theory.

Here, the shear stress criterion has been applied to these newer forms of construction, whereas normal practice has generally been—certainly for wire winding—to base design on tension in the layers.

Stresses caused by the shrinkage process before the pressure load comes on are often overlooked in design problems of this kind. Their evaluation, however, is simple (*15*) because they are merely the differences between stresses in the compound cylinder under the working pressure and those in a simple elastic cylinder of the same dimensions under the same pressure. This applies to direct as well as shear stresses. The limiting case where maximum residual shear stress just equals the maximum working stress is given by

$$\frac{n(K^{2/n} - 1)}{K^{2/n}} = \frac{2(K^2 - 1)}{K^2} \tag{22}$$

This is easily taken into account in design by constructing a diagram (Figure 4) in which a family of curves connecting P/S and K is plotted for different values of n with the curve of

$$\frac{P}{S} = \frac{2(K^2 - 1)}{K^2}$$

superimposed upon them. Then at any point in the shaded area of Figure 4, the residual shear stress exceeds the designed working stress, and any point on the boundary curve represents the condition that maximum shear stress under load is exactly equal and opposite to that existing when the load is removed. For any value of n greater than 2, there is a unique value of K that enables this condition to be achieved. When $n = 3$, this value is nearly 4.5; therefore, Figures 3, A, B, and C are for cylinders of that ratio and the value of P/S is then approximately 1.9. For larger numbers of components, the critical diameter ratio is less, and the resulting P/S is correspondingly reduced. These considerations are hardly worth pursuing from a practical point of view, but it seems reasonable to conclude that it is unwise to specify a value of P/S greater than about 1.8 for design purposes.

In practical shrink construction the effect of inevitable deviations from the exact dimensions specified, because of imperfections of machining and inspection, also have to be taken into account. This has been done (*13*) and with good modern workshop practice, it appears that maximum stress increases are unlikely to exceed 10%.

Reinforcement Methods

Reinforcement by strip and wire winding, and also by the multilayer process has been mentioned. Several other variations of these have been proposed, and the so-called "laminar construction" described by Birchall and Lake (*2*) comes into this category. All have the common feature of a central core tube but differ in the way this is reinforced. The core tubes are generally thin-walled by comparison with the complete cylinders; this is a valuable asset because they can be made of expensive corrosion resisting materials without seriously increasing the over-all cost. Another and less satisfactory common feature is the need to pay special attention to carrying axial forces.

Each method raises problems of its own, but there has been so much diversification that only an outline of the subject can be given here. On this basis, the laminar construction is a special case of elastic cylinders—or for higher pressures, possibly of autofrettage—but the others can be surveyed from the viewpoint of the shrink construction theory.

The wire-winding process was in regular use for many years for reinforcing large guns, but its great drawback was that the wire does nothing to help in carrying the longitudinal stresses. Nevertheless, where the core tube can do this safely, it is an easy and relatively cheap form of reinforcement. Newitt (*19*) has used it for small vessels carrying very high pressures. Usually, the design is based on having all the strands in equal tension when the internal pressure load is acting. Specifying this tension for each layer is complicated by the fact that outer layers, as they are wound on, tend to reduce tension in the inner ones.

Strip-winding is also a well established practice which, when the strip is flat, involves the same design considerations as wire-winding. In 1938, however, Schierenbeck (*21*) in Germany used a strip of W section and wound it so that each successive layer interlocked with the one below it. This has been successful, and it is even possible to thicken the ends with extra windings, and to drill and tap the resulting material for studs, as if it were an integral flange. Theoretical and practical aspects of this development have been dealt with by Siebel and Schwaigerer (*23*). Different materials for different layers can be used, and the process seems to have possibilities with the only condition being that all materials of construction must be weldable because each layer has to be anchored by welding.

One difficulty for the designer results from an interlocking effect which gives rise to longitudinal stresses of indeterminate magnitude and direction. However, the safest guide for the designer is that provided by the shrink construction theory—i.e., use a P/S ratio of 1.6 to 1.8, base the design only on windings, and ignore tube strength.

With multilayer construction more reliable data is available (*8*). Furthermore, as shown (*13*), the assemblies behave much as though they were built from separately machined tubes shrunk together, and yield at pressures only a few per cent below those calculated for ideal conditions. Attaching ends and, for long cylinders, making intermediate joints must be done by welding. This has been the occasional cause of weakness in the past, but presumably, these difficulties have been overcome by modern welding techniques.

The laminar construction (*2*) consists of reinforcing a thin core tube by slipping over it a large number of relatively thin rings. The end loads are then carried separately by an external portal frame, a procedure which could be applied equally well to a wire wound vessel. This method has been particularly useful for raising the pressure-carrying capacity of existing equipment. The strength of the assembly, so far as radial pressures are concerned, can be approximated by neglecting the core tube and treating the pile of rings as an elastic cylinder. The design of the portal frames can then be worked out according to the usual methods for such structures.

All these constructional methods and others of similar type have the advantage, where large sized cylinders are concerned, of avoiding heavy and costly forgings, and facilitating inspection. They are also generally cheaper than construction by shrinking a number of components, because the accurate machining and gaging of large components is avoided. For these reasons they are essentially processes for large-scale equipment.

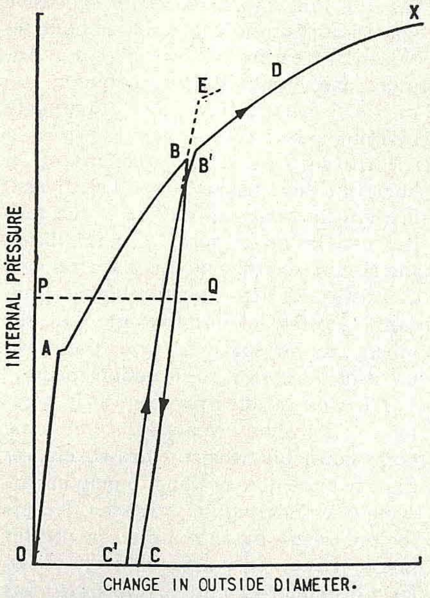


Figure 5. Internal applied pressure plotted against resulting changes in outside diameter

Ultimate Strength of Cylinders

Pressure required to burst a cylinder of ductile material is much greater than that which first causes overstrain (Figure 2). Furthermore, the thicker the cylinder, the greater this difference. Also, the cylinder swells considerably in girth before it fails, although its length changes little. Thus, radial and tangential strains are large but the longitudinal strain is small and, because metals in the plastic state deform without change of density, the cross-sectional area must remain substantially constant. Equation 5 shows that the whole problem can be solved if variation of shear stress across the wall is known, assuming it to be symmetrically distributed with respect to the central axis.

Because expansion is a constant area process, however, all strains across a section can be evaluated. The reasonable assumption (which the outcome seems to justify) that the relationship between shear stress and shear strain is always the same for a given material, no matter how the strains are caused—e.g., by tension, torsion, or bending—implies that the whole problem can be solved if the appropriate curve of shear stress and strain is available. This curve can most conveniently be derived from a torsion test. The actual process of derivation is given by Nadai (*18*) and the details of applying it to the cylinder problem are described by Manning (*12*). By thus computing pressures required to cause various assumed deformations, a maximum is obtained which must represent the ultimate bursting pressure of the cylinder.

Experimental determination of bursting pressures is more difficult than might be expected, mainly because the large deformation that must occur before the minimum bursting pressure conditions are reached takes considerable time. If pressure is raised too quickly, the specimen will burst at a pressure as much as 50% above that which would eventually cause failure if maintained for long enough. With care, however, consistent results can be obtained, which generally agree with the values derived in the way indicated here. This is shown by experimental work such as that of Faupel (*6*) and Crossland and Bones (*5*).

Study of cylinders by their shear stress and shear strain properties is of considerable value in dealing with autofrettage and creep. It should be emphasized, however, that these stresses and strains are assumed always symmetrical about the center, whereas it has been conclusively shown by Steele and Young (*25*) that this is seldom true for low carbon steels. Probably the same thing happens in any material in which there is an appreciable difference between the upper and lower yield stresses. This does not, however, affect the calculated bursting results, probably because when the whole wall is overstrained, the stress system again becomes symmetrical.

Autofrettage

The essential feature of this process is that it uses a part of the range between the overstraining and bursting pressures for the actual working pressure. It thus raises considerably the apparent strength of a simple cylinder without introducing the complications and cost of compound construction.

The process is best explained by reference to a typical example (Figure 5) where the internal applied pressure is plotted against the resulting changes in outside diameter. From O to A the whole system is elastic and the line is therefore straight; moreover, if the pressure is removed, the diameter will revert at once to its original size. At A, the elastic limit is reached at the inner surface, and the line bends over. If the pressure is reduced from a point such as B, the relation will follow a line BC which is more or less straight and parallel to OA, and when the pressure is reduced to zero, there will remain a permanent swelling in the diameter represented by OC. If the cylinder is then left for some time, it is probable that a slight contraction CC' will take place. On reapplying the pressure, the process will be represented by a line such as $C'B'$, again substantially straight, but enclosing a slight hysteresis loop. When the pressure approaches the highest value reached in the previous loading—i.e., that represented by the point B—the line curves over again and proceeds along almost exactly the course it would have taken if the rise of pressure had not been interrupted. A range such as $C'B'$ is sometimes referred to as induced elasticity.

This poses the question whether the working pressure could be raised to the value represented by say PQ? Although this is well beyond the elastic limit of the original cylinder, it is only about 60% of the induced elasticity and less than half the ultimate bursting pressure, X. The answer must lie in the stability and reproducibility of the system.

Although the idea of operating with overstrained material was propounded as long ago as 1906 by Malaval, a French artillery designer, the first account of its practical development seems to have been that by Langenberg (*9*) in 1925. Then in 1930 there followed Macrae's (*11*) well-known treatise. He investigated the effects of varying amounts of overstrain and of subsequent heat treatment at relatively low temperatures on the stability of specimens overstrained by different kinds of loading, with particular reference to cylinders under internal pressure. He concluded that suitable heat treatment would remove any tendency for elastic afterworking—i.e., partial recovery as represented by CC' in Figure 5—and would eliminate the hysteresis between descending and ascending curves and make them straight lines. He further suggested that after heat treatment, a line such as $C'B'$ would continue straight to a pressure substantially greater than B, and he called this increase the elastic gain (Figure 5, dotted line to E).

The first essential to be remembered in considering the application of autofrettage is that the material overstrains symmetrically. In practice, this means that a curve like that shown in Figure 5 must be obtained on each diameter. This is a considerable limitation, but it does not affect the use of metals such as gun steels which are the usual choice for high pressure equipment. There is also some doubt about the possibility of entirely removing the hysteresis between successive applications of pressure. In fact its persistence is expected if shear stresses in the inner layers are considered. As the overstraining pressure comes on, this stress-strain curve runs similar to line $OABE$ in Figure 6. If the pressure is taken up to E and then released, the line will first run back to F, but because the cylinder has suffered a permanent swelling, the still elastic outer layers will force the overstrained material into compression and so cause the shear stress to change sign, and the line will continue along the straight course to G. Then however, it will bend over until it reaches H, where HH' is the residual shear strain of the unloaded cylinder. When pressure is reapplied, the shear curve will begin from H and run through J to K where it will bend towards E. In accordance with the well known Bauschinger effect, the straight line range of the reversed stress, FG, will be much shorter than the recovery line, EF.

It may be possible to eliminate the hysteresis if the pressure range is much less; for instance the recovery line, BC, in Figure 6 is such that it is only necessary to reverse the stress to the value given by D, where CD is still a straight line. Reapplication of pressure should then cause the shear stress to return along the straight line to B.

The question whether a certain amount of hysteresis can be tolerated and if so, how much, is difficult for designers. In general, cylinders not subjected to frequent cycles of pressure fluctuation can be safely operated with the zone of overstrained material extending half way through the wall. For pump and compressor cylinders, on the other hand, the situation is more difficult and very little experimental data is available. It is suggested that overstraining in such instances be strictly limited.

Determining state of stress and strain in the wall of a cylinder subjected to autofrettage is rather complicated if a rigorous analysis is demanded. Various solutions have been published, but those of Hill, Lee, and Tupper (*7*), and of MacGregor, Coffin, and Fisher (*10*) are perhaps the best known. A graphical approximation developed by Manning (*14*) is probably accurate enough for most practical problems. It involves the preparation of a chart which is really the plot of the shear stress, the radial stress, and the function, u/r, against the logarithm of the unstrained radius for a thick cylinder, say for a diameter ratio of 10 to 1, which is partially overstrained by internal pressure. Suppose that the boundary between the elastic and plastic regions is taken at a radius of 4 times the bore; the elastic part of the diagram is easily computed by the ordinary Lamé analysis, knowing that the shear stress at the boundary has its limiting elastic value. The portion of the cylinder within the boundary is plastic and the stresses there must be computed from the shear strains in the manner already indicated. Thus, the area under the shear stress curve between any two ordinates represents the difference in radial stress between the appropriate radii.

The application of this procedure to a design problem is best illustrated by an example. Figure 7 is the chart for a chrome-molybdenum steel of 180,000 pounds per sq. inch ultimate tensile strength, and when holding a pressure of about 350,000 pounds per sq. inch the walls of this 10 to 1 cylinder have been overstrained outwards to a radius of about 4.2 times the bore, the material outside that being still elastic. It is also possible to follow what happens with cylinders of smaller diameter ratio. For instance, the intercept LK represents a diameter ratio of 2 to 1 and the corresponding part of the radial stress curve, AB, would apply to a cylinder of this ratio, carrying an internal pressure of 135,000 and an ex-

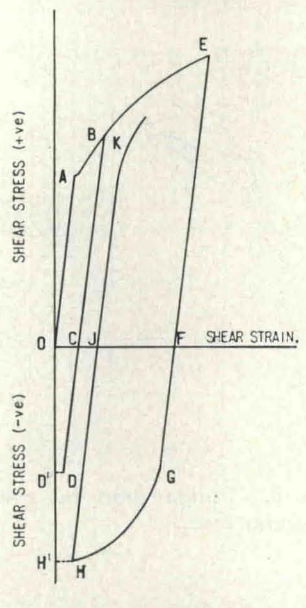


Figure 6. Shear stress vs. shear strain in inner layers of overstrained cylinders

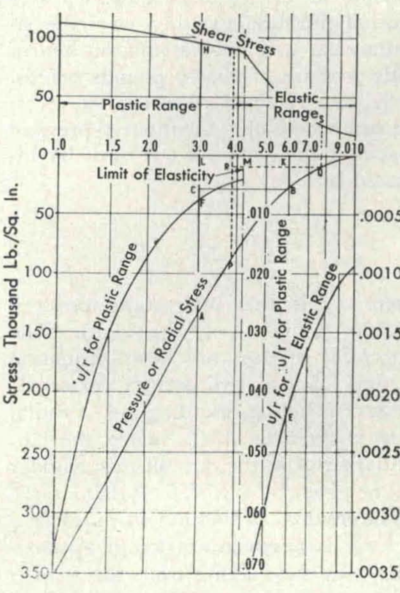


Figure 7. Design chart for autofrettage cylinders

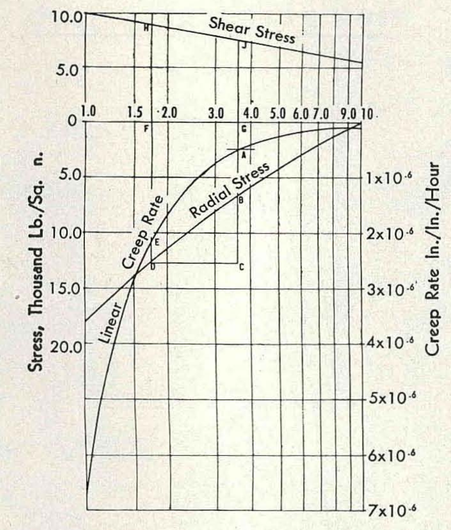


Figure 8. Design chart for cylinder undergoing creep

ternal pressure of 28,000 pounds per sq. inch. The resulting shear stress distribution is given by *HG.* and the deformations resulting while these pressures act are given by the corresponding curves, the points *E* and *F* representing the values of u/r for the outside and inside surfaces, respectively. Because of the rapid change of the ratio, u/r, with radius, the curve for this is split into two parts with different scales.

The part of the cylinder which will be overstrained by these pressures is represented by *LM*, while *MK* remains elastic. The same amount of overstraining will however be occasioned by an internal pressure of 107,000 pounds per sq. inch with no external pressure; shear stress will also be the same, but deformations must be corrected because the values represented by *E* and *F* correspond to the strains caused by this internal pressure, together with the simultaneous action of a uniform hydrostatic pressure of 28,000 pounds per sq. inch. On removal of this latter, as in the case where only the internal pressure is acting, the value of u/r must be increased by

$$\frac{P_o}{E}(1 - 2c) \qquad (23)$$

where P_o is this hydrostatic pressure (28,000 pounds per sq. inch). The tangential stresses are from Equation 4, and if no hydrostatic (external) pressure is acting, the diagram of radial stress is given by *ABC;* hence. the tangential stress at the bore of this cylinder will be given by $2 \times HL - AC$.

The residual stresses left when internal pressure is removed are found approximately by subtracting from the stresses computed in the preceding paragraph, those that would result in a similar elastic cylinder holding the same pressure, assuming it could do so without overstrain.

The convenience of the logarithmic scale for the radii is now seen. Thus, to contain 80,000 pounds per sq. inch with a cylinder having a diameter ratio of 2, the procedure is to find an intercept of 2 on the horizontal scale which corresponds with an intercept of 80,000 on the vertical. The dotted lines *PRTSQ* on the diagram show this; the region of overstrain in this case only extends to about 4.2/3.8, or for some 10% of the wall thickness.

The practical results of this method, as applied to design and construction of laboratory scale apparatus, have been satisfactory, and much time and labor can be saved in the early stages of a high pressure project by investigating it in this way.

Creep

Where cylindrical vessels and pipes are operated at temperatures sufficiently high to produce creep, design problems become more involved. Fortunately, however, they are capable of a restricted solution which is sufficient for most practical cases. In what follows only secondary creep—i.e., creep which proceeds at a constant rate—is dealt with, and it is assumed also that the temperatures do not change, although they need not be the same at all points in the cylinder wall. Further assumptions are that the working pressure is maintained steadily, and that the stresses, strains, and temperatures are always symmetrically distributed with respect to the axis.

It is generally agreed, on the basis of reliable experiments, that any given stress system will always produce the same rates of creep in the same material. For the solution of the present problem, however, it is necessary to postulate that the converse is also true—namely, that where a particular rate of straining is imposed on a material. the same stresses will always be induced in it. It is difficult to see how this can be otherwise, but its assumption often causes surprise. It appears to have been suggested originally by Bailey (*7*), who also developed and tested the theory based upon it.

Experiment shows that creep in a cylinder proceeds mainly in the cross section, the longitudinal strains being always small compared with those in the radial and tangential directions. Thus, creeping of a cylindrical wall under the influence of internal pressure is virtually a constant area process, and the rates of creep at all points in the wall can therefore be expressed in terms of the creep at any one point, say the bore, by simple geometry. If then the appropriate relation between shear stress and creep rate is available, the shear stress distribution across the wall can be derived, and thence by integration as in Equation 5, the presssure needed to give this rate of creep in the bore. Some trial and error may still be needed to deal with a particular case, but the problem is thus essentially solved.

On the assumption that rate of shear strain is governed by the maximum shear stress, the relation between shear stress and shear creep rate is easily derived from tensile creep rate. Then, if the tangential strain is e_t and the radial strain e_r, the condition of constant area demands that

$$(1 + e_t)(1 + e_r) - 1 = 0 \qquad (23)$$

and, since the strains are always small in practical applications

$$e_t = -e_r \qquad (24)$$

and the shear strain q is given by

$$q = 2\, e_t \qquad (25)$$

Now if the radial shift after any time, t, is u, the condition of constant area demands that

$$u_o r_o = ur = u_i r_i \qquad (26)$$

to a reasonable degree of approximation. Moreover, since $e_t = u/r$, it is evident that

$$q = \frac{2u_i r_i}{r^2} = \frac{2u_o r_o}{r^2} \qquad (27)$$

Differentiating with respect to time, keeping r constant then gives

$$\left(\frac{\partial q}{\partial t}\right)_r = \frac{2r_i}{r^2}\left(\frac{\partial u_i}{\partial t}\right)_r = \frac{2r_o}{r^2}\left(\frac{\partial u_o}{\partial t}\right)_r \qquad (28)$$

These relations together with the appropriate creep data permit the construction of a chart which will enable most problems of creep in a cylinder of the particular material to be quickly solved. Figure 8 is an example derived from the tensile data of a stainless steel containing approximately 18% chromium, 8% nickel, and about 1% of titanium, creeping at 600° C. Like Figure 7, it represents a cylinder with its outside diameter 10 times the bore, and Equation 28 then shows that the creep rate at the bore will be 100 times what it is at the outside. A shear strain rate of 10^{-7} at the outside surface has been assumed, and this is equivalent to a linear strain rate of 6.67×10^{-8}. The strain rates at all the other points can now be calculated; thence, having prepared a shear stress *vs.* shear strain rate diagram from the tensile data, the appropriate shear stresses can be found. (In deducing the relation between shear stress and shear strain rate from tensile data it is possible, as Shepherd (*22*) has shown, to take into account the third principle stress as is done in the von Mises hypothesis. The equivalent shear stress is then $1/\sqrt{3}$ times the tensile stress.) Again, since the radii are plotted on a logarithmic scale, the area under the shear stress curve gives the radial stress.

The usefulness of such a chart can be illustrated by an example. Suppose that a pressure of 6000 pounds per sq. inch at 600° C. has to be contained with this material, and that the rate of creep on the outside surface must not exceed 5×10^{-7} inch per inch per hour. The chart shows that this creep rate is represented by point *A* which corresponds to a radial stress represented by *B*. A perpendicular *BC* equal to the required pressure is then drawn, and a horizontal line through *C* to cut the stress curve again at *D*. The intercept *CD* gives the required diameter ratio which is just over 2 to 1.

The rate of creep at the bore surface (point E) is 2.2×10^{-6}, a value which might be considered too high. In a year, the bore diameter would have increased by nearly 2% and the walls would have become $1^1/_2$% thinner, which in turn would mean that for the same internal pressure, the stresses would rise and consequently also the creep rates. In fact, the whole process is one of gradually increasing rate in the direction of ultimate failure. This can be seen from the shear stresses represented by the portion *HJ* of the shear curve. The progress of creep as viewed on this chart, will consist of a movement of the ordinates *HF* and *JG* to the left in such a way that area *HFGJ* remains constant.

Opinion still differs as to the best design criterion for thick cylinders subjected to creep conditions, but the chart should suffice for any of the normal design theories because all the stresses are obtainable from it. The radial and shear stresses can be read off directly, and the tangential stress is obtained from Equation 4, the actual stress at the bore in the example being given by $2 \times HF - BC$. This could be used, for instance, with a time-rupture curve.

For heat flow and temperature gradients the procedure is more involved. A set of curves relating shear stress and temperature for a number of constant creep rates has first to be drawn. From these, the shear stress is found at each point because creep rate is determined by the geometry of the system, and the temperature by the rate of heat flow and the thermal conductivity of the material. A shear stress curve can thus be obtained which on integration gives the radial stress. Some trial and error is necessary, but the method is straightforward.

Fatigue

Published data on fatigue resistance of the commoner engineering materials is immense, but unfortunately most of it relates to the simpler kinds of loading such as reversed bending. Even when more complex systems of stress have been employed, one of the principal stresses is usually either zero or negligibly small, and few of the results are therefore applicable directly to the design of high pressure cylinders.

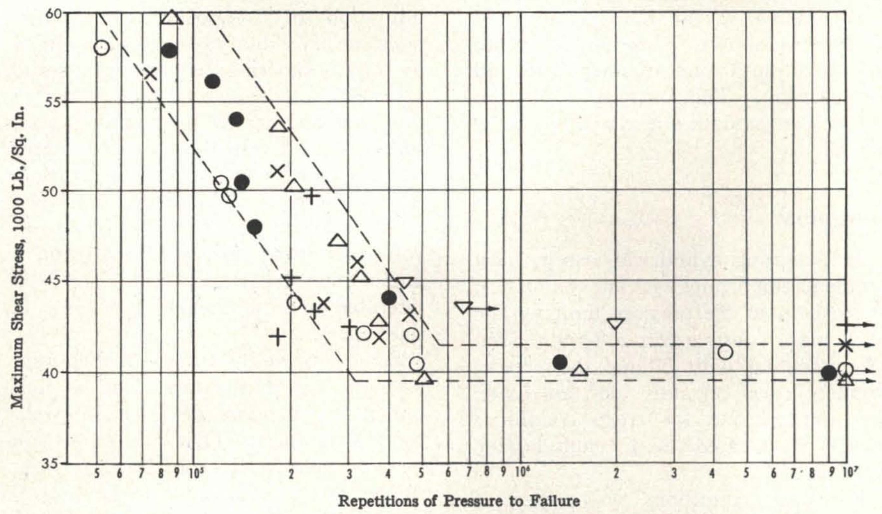


Figure 9. Fatigue of nickel-chromium-molybdenum steel cylinders *(17)*

The problem is undoubtedly serious, because material that forms the walls of pump and compressor cylinders is subjected to constant repetitions of pressure while in service, and few engineers concerned with the operation and maintenance of such equipment have not experienced a serious breakdown due to fatigue. It is true that these troubles usually take place in the more complicated shapes like cylinder heads and valve passages, but even the plain cylindrical walls provide their share of sudden and unpredictable failures.

So far the only published results found are those of Morrison and others *(17, 20)* at the Bristol University Engineering Laboratories in England. In this work, cylindrical specimens of either 0.6- or 1-inch bore with various diameter ratios between 1.2 and 3.0 were tested by reciprocating within them a plunger, the internal space being filled with oil. With one material for which tests have been reported, the results seem remarkably consistent. This was an oil-hardened and tempered alloy steel containing approximately 0.3% carbon, 2.5% nickel, 0.6% chromium, and 0.6% molybdenum with an ultimate tensile strength of 124,000 pounds per sq. inch; its limiting endurance when subjected to reversed torsion was ±43,500 pounds per sq. inch shear stress. Tests with cylinders under repetitions of pressure have shown relatively little scatter when the maximum shear stress is plotted against the number of repetitions to failure, but the maximum direct stress does not give nearly such consistent results. There is a well-defined fatigue limit at a shear stress of about 40,000 pounds per sq. inch (Figure 9) but the stress range is from 0 to that figure, whereas in the torsion tests the range of shear stress is 87,000 pounds per sq. inch. A wider range of endurance when the loading reverses itself—i.e., when the mean stress is zero—would be expected, but hardly by a factor of more than 2 as was found in these results.

It appears that the anomaly results partly from the particular type of stress distribution involved, and partly from the effect of fluid pressure on a highly stressed surface. In fact, as Parry *(20)* has shown, a considerable raising of the endurance limit appears to result from protecting the inner surface with an impermeable nonmetallic layer; in some cases, the increase was as much as 40%. Similar effects were also obtained by honing the bore immediately before testing. Since the specimens plotted in Figure 9 were all vacuum-annealed before testing, the slight cold-working introduced by honing, which would have been removed by the annealing, must have had a beneficial effect.

Morrison has found that a 3% chrome steel of roughly the same static properties has given a similar endurance limit, and preliminary tests with a low-carbon mild steel suggest that its endurance as a cylinder will also be at a range of shear stress only about half that endured in reversed torsion.

The cracks resulting from these tests were characteristic and different from those produced by static conditions of loading. In the latter, the fissure nearly always takes a spiral course across the wall, and it is preceded by considerable swelling and deformation *(6)*. The fatigue cracks, on the other hand, were radial and appeared without warning. The development of this type of failure was also interesting: Usually several separate cracks started at different points on the bore surface and spread gradually in planes parallel with the axis, until one of them reached the outside. The surfaces of these cracks were smooth and characteristic of fatigue.

Summary

In designing cylindrical vessels, most problems can be met with fair confidence, at least up to the pressure limit for the particular construction offered. But when dealing with fatigue the present resources are certainly not adequate, and the solution for creep conditions must be treated with considerable caution.

For static conditions where there is no risk of creep, simple elastic cylinders can be made from material having a strength up to about 180,000 pounds per sq. inch ultimate tension, although this may be attainable only in comparatively small pieces of metal. With good grade alloy steels, however, it should be approachable, and the material should still retain some ductility and shock resistance. It should be possible to operate with suitable safeguards to a pressure within a few per cent of that which causes the beginnings of overstrain. Also, if it is assumed that a diameter ratio of 3 to 1—i.e., wall thickness equal to bore diameter—is about the limit of reasonable proportions, the pressure ceiling for a simple elastic vessel is established at about 75,000 pounds per sq. inch, or about 5000 atm.

Using compound construction, the diameter ratio can be increased, probably to about 4.5 to 1. Ratio of pressure to maximum shear stress can then be raised to about 1.9, but some safety factor is desirable and a working shear stress of 70,000 pounds per sq. inch seems a reasonable limit; the maximum working pressure could then be raised to 133,000 pounds or about 9000 atm. Autofrettage further extends the range without seriously increasing the uncertainties and a pressure of 220,000 pounds with a 4.5 to 1 cylinder should be possible, the region of overstrain then extending roughly to the geometric mean radius. That is, however, within 75% of the ultimate bursting pressure. Normally, autofrettage in industry is not operated right up to the originally applied pressure but to about 80% of it; this would give a safe working pressure of about 12,000 atm.

These figures can perhaps be exceeded to a small extent—e.g., larger diameter ratios are possible, especially with some techniques of compound construction. But the limit for the condition of indefinite durability can hardly exceed about 15,000 atm. or about 225,000 pounds per sq. inch. To exceed this, special techniques are presumably needed, such as those used by Bridgman (*3*) where limitation of compound construction from rise of residual stresses is surmounted by slightly tapering the components and arranging for them to be pushed further into one another as the working pressure comes on. In this way, the degree of shrinkage is increased while raising the pressure, and similarly decreased while lowering it. The problems involved in using this, particularly on a larger than laboratory scale, are considerable and lie outside the scope of this report.

Steady pressures up to 15,000 atm. at moderate temperatures can be contained in cylinders designed by well-tried techniques. That this range can be very greatly exceeded is clear from the recent announcements by the General Electric Co. and by the Allmänna Svenska Elektriska Aktiebolaget (A.S.E.A.) in Sweden of the synthesis of diamonds. On the other hand, possibilities for the chemical industry within the range considered here are considerable and little explored.

The increased capital outlay required is often given as a reason for not pursuing schemes involving high pressures. Some increase in cost, of course, is unavoidable, but this is partly offset by such factors as smaller reaction volumes needed, higher yields, and quicker reactions. For example, when the Ziegler low pressure process for making polyethylene was introduced, it was suggested in some quarters that the product would be so much cheaper that the high pressure plants would have to go out of business. Nothing of the sort has happened, however, and it appears that the high pressure process is at least as economic as any of its more recent low pressure competitors. This example could be typical of other processes, and high pressure techniques are certain to assume increasing importance for the chemical industry.

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Nomenclature

P = applied pressure
f = direct stress
e = direct strain
S = shear stress
q = shear strain
r = radius
D = diameter
t = thickness of cylinder wall; also time in differential coefficients
L = over-all length of cylinder
u = radial shift
v = shrinkage
x, y = increase in external diameter and length, respectively
K = over-all diameter ratio
k_1, k_2, etc. = diameter ratio of components
λ = safety factor
n = number of components
E = Young's modulus
c = Poisson's ratio

Subscripts

i = internal
o = external
t = tangential
r = radial
a = axial

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